

Multi-cell Downlink Dimensioning in NB-IoT Networks

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Abstract—This work derives a numerical dimensioning model for the downlink of a multi-cell Narrow-band Internet of things (NB-IoT) network. In such a system, the number of resource blocks (RBs) that a user requires depends on the SINR value which is a function of user position and the fading model. A network outage happens when the number of resource blocks available in the network is smaller than the one required by the user. In this paper, we propose a model to calculate the number of resource blocks needed in the NB-IoT network to guarantee the target outage probability, taking into account the repeated transmission feature of NB-IoT.

I. INTRODUCTION

The Internet of Things (IoT) has been in popularity in recent years. The number of IoT devices is increasing dramatically both in the number and the area of applications (health-care, logistic, agriculture, transportation, etc). IoT devices have strong power consumption, deep coverage, and low device complexity requirement. Besides, the number of devices in IoT networks which is several thousand times greater than in Long Term Evolution (LTE) networks is a challenge in the network design. NB-IoT is a Low Power Wide Area (LPWA) technology, which was standardized in the Third Generation Partnership project (3GPP) release, specifies a new random access (RA) procedure and a new transmission scheme for IoT [1]. The advantages of NB-IoT network is providing deep coverage, low power consumption and support of huge number of connection. In our work, we focus to investigate the radio resource allocation in the downlink of multi-cell NB-IoT.

Resource allocation in NB-IoT network is based on the resource block. The amount of resource blocks allocated to a sensor node can be calculated from the channel quality and the data rate required by the application. If the total number of resource blocks required is greater than the cell capacity, packets will be lost. One of the main objectives of Radio Resource Allocation in such a system is to determine the radio resources in the network to guarantee the required Quality of Service (QoS), IoT device density and eNodeB (eNB) density. This work proposes an analytical model that calculates the required number of resource blocks as a function of the network parameters in multi-cell downlink NB-IoT. In this network, we consider that the distribution of users and eNBs follows a random Poisson Point Process (PPP).

Poisson Point Process has been widely used in the literature [4] - [9] to model the positions of eNBs and sensor users in the cellular network. The problem of Orthogonal

Frequency-Division Multiple Access (OFDMA) based the network planning and dimensioning have been more recently under investigation. In [7], the authors give robust methods to compute the resource outage probability upper-bound for LTE network.

In [8], the authors propose an uplink NB-IoT dimensioning model in which the interference is considered. However, the proposed model is performed only on single-cell with a single class of user.

The main contribution of our work is the derivation of a numerical model to calculate the required radio resource blocks in the multi-cell downlink NB-IoT network using the Bell polynomials. This model is very necessary for operators because it shows how to manage the available spectrum resource. Based on the requested number of resource blocks, we determine the network outage probability depending on the transmission rate of sensors, the path-loss attenuation and the densities of IoT devices and eNBs.

This paper is structured as follows. We introduce the system model of the NB-IoT system and set up the problem of radio resource allocation in section II. In Section III, we consider the outage probability and derive the Bell polynomials method to find the number of resource blocks necessary to achieve a target outage probability. We then present numerical results to estimate the outage probability and analyze the relation between the required number of RBs and the IoT device density in Section IV. Finally, conclusions are drawn in Section V.

II. SYSTEM MODEL

In this section, we consider the performance of the downlink channel in NB-IoT network. Networks are deployed according to a Poisson point process. Since we study an NB-IoT system, the downlink channel of each antenna is divided into subcarriers of 15 kHz. Assume that each cell uses a different frequency band from its neighboring cells and has M available resource blocks. These resource blocks are then allocated to the active users, whose positions are drawn according to a Poisson point process.

A. Sensor users model

We suppose that the positions of sensor users in the cell follow a spatial PPP Φ_u with intensity λ_u . In a given coverage area, the number of sensors follows a Poisson distribution

while their locations follow uniformly distribution. From now on, the two terms of intensity and density are used interchangeably. Let Φ_b be a point process of intensity λ_b , that represents the set of eNBs. Therefore, the average number of sensors in a cell inside the coverage area, denoted by u , can be calculate by

$$u = \lambda_u S_{area} \quad (1)$$

where S_{area} is the area of the selected cell.

B. Network model

We assume that each sensor user is connected to the nearest eNB. The cell boundaries form a Voronoi tessellation on the plane, as illustrated in Figure 1.

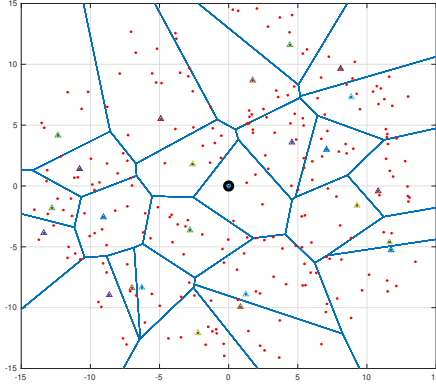


Figure 1. A realization of the network modeled by a PPP Φ_b and Φ_u . The typical eNB is denoted by black dot when the sensors by red small dots.

Considering a typical eNB at the origin O . The received power by a sensor located at distance x from O is $P_t x^{-\alpha}/a$, where P_t is the transmitted power from eNB, a is the propagation constant and α is the path loss exponent. We assume that each eNB allocates Physical Resource Blocks (PRBs) to its sensor users and each PRB has a bandwidth denoted by W_{rb} .

Depending on the transmission rate and the position of the sensor user, each user has a given number n PRBs. In this work, we denote a required transmission rate by C^* .

The sensor user located at distance x from O is able to decode the signal only if the Signal to Interference plus Noise Ratio (SINR) $\text{SINR}(x) = \frac{P_t x^{-\alpha}/a}{I + \sigma^2}$ is above a threshold θ_{thr} , where I is the received co-channel interference and is assumed to be negligible in our analysis. σ^2 is the thermal noise power. If the SINR is below the critical threshold, then the sensor is said to be in outage and cannot proceed with its communication.

The corresponding maximum coupling loss (MCL) can be computed based on the received SINR. The relationship between SINR and MCL is given as follow

$$\text{SINR}_{target} = P_t + 174 - \text{NF} - 10\log_{10}(\text{BW}) - \text{MCL}, \quad (2)$$

where NF is the receiver noise figure and BW is the allocated bandwidth. For the simulation, the same information is repeated n_{rep} times where n_{rep} being the number of repetitions. The number of repetitions is computed based on different MCL values which are presented in Table I [12].

MCL	Repetition
Below 145 dB	1
146 to 148 dB	2
149 to 151 dB	4
152 to 154 dB	8
155 to 157 dB	16
158 to 160 dB	32
161 to 163 dB	64
Above 164 dB	128

Table I
COVERAGE CLASSES WITH REPETITION FACTOR [12].

According to Table I, there are 8 classes of users having the repetition factors varying from 1 to 128. We assume that each sensor node which has the same number of repetitions attempts to transmit with the same probability $p_l = 2^l/128$, where $l = (0, 1, \dots, 7)$ represents the class of user corresponding to the repetition factor.

Definition 1: (Number of resource block). The eNB may allocate a maximum number N_{max} of resource blocks to each sensor node at each time slot. According to the Shannon formula, a sensor located at a distance x from the center O demanding service of data rate C^* , has the number of PRBs which is defined as follows:

$$N(x) = \min \left(\left\lceil \frac{C^*}{W_{rb} \log_2(1 + \text{SINR}(x))} \right\rceil, N_{max} \right) \quad (3)$$

$$= \min \left(\left\lceil \frac{C^*}{C(x)} \right\rceil, N_{max} \right)$$

where $\lceil y \rceil$ is the ceiling function of y , W_{rb} is the bandwidth of a resource block, the quantity N_{max} is the maximum number of resource blocks that can be allocated to one sensor user and $C(x)$ is the throughput of a sensor located at a distance x from O .

From (3), we realize that the sensor in which a bad channel quality needs a higher number of PRBs to achieve the transmission rate C^* . Let d_n be the distance from O that verifies, for all $x \in (d_{n-1}, d_n]$, $N(x) = n$, with

$$n = \frac{C^*}{C(d_n)} \quad (4)$$

is an integer and

$$d_n = \begin{cases} 0 & \text{if } n = 0 \\ \left[\frac{a(I + \sigma^2)}{P} \left(2^{\frac{C^*}{n W_{rb}}} - 1 \right) \right]^{-\frac{1}{\alpha}} & \text{otherwise.} \end{cases} \quad (5)$$

From (4), the area of typical cell O can be divided into rings with radius d_n ($1 \leq n \leq N_{max}$, $0 \leq d_{n-1} \leq d_n$). When the sensor requests n PRBs to achieve the transmission rate C^* , the sensor is said to be located in the area between two rings

of radius d_n and d_{n-1} . If $x > d_{N_{max}}$, we consider a sensor is out of service and $n = 0$.

In addition, from (2) and Table I, we realize that the number of repetitions of each user depends on its SINR value. Let r_l be the distance from O , the typical cell can be divided into rings with radius r_l ($0 \leq l \leq 7, 0 \leq r_{l-1} \leq r_l$). The area between the rings of radius r_l and r_{l-1} defines the region of the cell where the sensors have $n_{rep}(l)$ repetitions ($n_{rep}(l) = 2^l, l = 0, 1, \dots, 7$). For each user having a distance $x \in (r_{l-1}, r_l]$, it has the number of repetitions $n_{rep}(l)$ with:

$$r_l = \begin{cases} 0 & \text{if } n = 0 \\ \left[\frac{a(I+\sigma^2)}{P} \text{SINR}_{\text{target}}(l) \right]^{-\frac{1}{\alpha}} & \text{otherwise.} \end{cases} \quad (6)$$

C. The total number of requested PRBs

We propose a definition of "virtual number of user", which is equal to the number of users multiplied by its number of repetitions. To qualify the "virtual number of users", which have n PRBs and n_{rep} repetitions, we consider 8 independent Poisson random variables denoted by $X_n^{n_{rep}(l)}$, with parameter $\mu_n^{n_{rep}(l)}$. Note that the sensor users in the same repetition area have the same transmit probability $p_l = 2^l/128$. The mean number of users μ_n that locate between two rings $B(0, d_n)$ and $B(0, d_{n-1})$ is then computed as

$$\mu_n = \sum_{l=0}^7 \mu_n^{n_{rep}(l)}, \quad (7)$$

where

$$\begin{aligned} \mu_n^{n_{rep}(l)} &= n_{rep}(l) p_l \lambda \pi \xi \eta \\ \xi &= d_n^2 \mathbb{1}_{\{r_{l+1} \geq d_n\}} + r_{l+1}^2 \mathbb{1}_{\{r_{l+1} < d_n\}} \\ &\quad - (r_l^2 \mathbb{1}_{\{r_l \geq d_{n-1}\}} + d_{n-1}^2 \mathbb{1}_{\{r_l < d_{n-1}\}}) \\ \eta &= \begin{cases} 0 & \text{if } B < 0 \\ 1 & \text{if } B > 0 \end{cases} \end{aligned} \quad (8)$$

The number of users that lie between two rings $B(0, d_n)$ and $B(0, d_{n-1})$ is a Poisson random variable denoted by V_n :

$$V_n = \sum_{l=0}^7 X_n^{n_{rep}(l)} \quad (9)$$

Finally, the total number of required PRBs in the typical cell is defined as the sum of demanded PRBs by sensor users in each ring of radius d_n and can be represented as:

$$\Gamma = \sum_{n=1}^{N_{max}} n V_n \quad (10)$$

The random variable Γ which is named compound Poisson sum is the sum of weighted Poisson random variables.

III. THE OUTAGE PROBABILITY

The outage probability is defined as the probability in which the number of the total request PRBs in the cell is greater than a threshold value fixed by the operator $P_{out} = P(\Gamma \geq$

$M)$, where M is the output number of PRBs that required to guarantee a predefined quality of services.

$$\mathbb{P}(\Gamma \geq M) = \mathbb{E}_{\Phi_b}[\mathbb{P}(\Gamma \geq M | \Phi_b)] \quad (11)$$

In this work, we derive a mathematical model called the exponential Bell polynomials [14], [15] to calculate the outage probability. This model is widely used in order to evaluate some integrals and alternating sums. In the following subsection, we recall the definition and some properties of Bell Polynomials.

A. The exponential Bell Polynomials

The exponential complete Bell polynomials B_p are defined by

$$B_p(x_1, x_2, \dots, x_p) = \sum_{k_1+2k_2+\dots=p} \frac{p!}{k_1!k_2!\dots} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \dots \quad (12)$$

and verify the following formula given by the generating function

$$e^{\sum_{j=1}^{+\infty} x_j \frac{t^j}{j!}} = \sum_{p=0}^{+\infty} \frac{t^p}{p!} B_p(x_1, x_2, \dots, x_p) \quad (13)$$

Also, if we consider the following matrix $A_p = (a_{i,j})_{1 \leq i,j \leq p}$ defined by

$$\begin{cases} a_{i,j} = \binom{p-1}{j-i} x_{j-i+1} & \text{if } i \leq j, \\ a_{i,i-1} = -1 & \text{if } i \leq 2, \\ a_{i,j} = 0 & \text{if } i \geq j+2, \end{cases} \quad (14)$$

such that

$$A_p = \begin{bmatrix} x_1 & \binom{p-1}{1} x_2 & \binom{p-1}{2} x_3 & \binom{p-1}{3} x_4 & \dots & x_p \\ -1 & x_1 & \binom{p-2}{1} x_2 & \binom{p-2}{2} x_3 & \dots & x_{p-1} \\ 0 & -1 & x_1 & \binom{p-3}{1} x_2 & \dots & x_{p-2} \\ 0 & 0 & -1 & x_1 & \dots & x_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 & x_1 \end{bmatrix}$$

then the complete exponential Bell polynomial $B_p(x_1, \dots, x_p)$ can be defined as the determinant of this matrix.

$$B_p(x_1, \dots, x_p) = \det(A_p) \quad (15)$$

Based on the exponential complete Bell polynomials, we can give the expression of the outage probability as the following proposition.

Proposition 1 Let Λ be a random variable such that $\Lambda = \sum_{n=1}^N n V_n$, with V_n are Poisson random variables of intensity w_n . Let x_j be defined as

$$x_j = \begin{cases} w_j j! & \text{if } 1 \leq j \leq N \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

The probability that Λ exceeds a threshold M can be expressed as a function of the exponential complete Bell polynomials by

$$\mathbb{P}(\Lambda \geq M) = 1 - H \sum_{k=0}^{M-1} \frac{B_k(x_1, \dots, x_k)}{k!} \quad (17)$$

where $H = e^{-\sum_{n=1}^N w_n}$. The proof is presented in Appendix.

B. The outage probability

We apply proposition 1 with a Poisson random variable V_n of a parameter $w_n = \mu_n$. The outage probability conditionally on Φ_b in (11) can be calculated by averaging over the realization of Φ_b :

$$\mathbb{P}(\Gamma \geq M) = \mathbb{E}_{\Phi_b} \left[1 - H \sum_{k=0}^{M-1} \frac{1}{k!} B_k(x_1(\Phi_b), \dots, x_k(\Phi_b)) \right] \quad (18)$$

where $x_j = \mu_j j!$ and $H = e^{-\sum_{n=1}^{N_{max}} \mu_n}$.

As we mentioned in section II, μ_j is the mean number of sensor users that requested j PRBs with $1 \leq j \leq N_{max}$. In case $N_{max} < j \leq M$ we assume that sensor user j is out of service and $\mu_j = 0$. The outage probability given by equation (18) becomes:

$$\mathbb{P}(\Gamma \geq M) = 1 - \left(\hat{B}_0^{(N)} + \hat{B}_1^{(N)} + \dots + \frac{1}{(M-1)!} \hat{B}_{M-1}^{(N)} \right) \quad (19)$$

where

$$\begin{aligned} \hat{B}_0^{(N)} &= \frac{1}{N} \sum_{i=1}^N H^{(i)} \\ \hat{B}_1^{(N)} &= \frac{1}{N} \sum_{i=1}^N H^{(i)} x_1^{(i)} \\ \hat{B}_2^{(N)} &= \frac{1}{N} \sum_{i=1}^N H^{(i)} (x_1^{(i)2} + x_2^{(i)}) \\ \hat{B}_3^{(N)} &= \frac{1}{N} \sum_{i=1}^N H^{(i)} (x_1^{(i)3} + 3x_1^{(i)} x_2^{(i)} + x_3^{(i)}) \\ \hat{B}_4^{(N)} &= \frac{1}{N} \sum_{i=1}^N H^{(i)} (x_1^{(i)4} + 6x_1^{(i)} x_2^{(i)} + 4x_1^{(i)} x_3^{(i)} + 3x_2^{(i)2} + x_4^{(i)}) \\ &\vdots \end{aligned} \quad (20)$$

Note that for $j > N_{max}$, the value $x_j = 0$.

IV. NUMERICAL ANALYSIS

Numerical analysis is performed on NB-IoT networks which are generated according to a Poisson Point Process. We explore three scenarios as follows

- 1) for a given user density λ_u and the transmission rate C , the outage probability is derived in a function of the requested PRBs fixed by the eNB M ;
- 2) for a given eNB density λ_b and the transmission rate C , the outage probability is derived in a function of the requested PRBs fixed by the eNB M ;
- 3) for a given outage probability P_{out} , the required number of available PRBs in the network is derived in function of the sensor user intensity λ_u ;

The proposed model estimates the outage probability in the multi-cell downlink NB-IoT. For numerical purpose, we consider an area of radius $R = 15$ km and each eNB transmits

the same power of $P = 40$ dBm. The sum of the downlink thermal noise power and the receiver noise figure is set to $\sigma^2 = -129,2$ dBm. For our network model, the propagation parameter is $a = 130$ dB and the path loss exponent is considered to be $\alpha = 3.5$. In this work, we consider only one class of service with $C = 80$ kbps.

Figure 2 illustrates the described model in MATLAB for the eNB's density $\lambda_b = 0.9$ eNBs/km² and two values of the active sensor user's density $\lambda_u = 15$ users/km², $\lambda_u = 20$ users/km². We observe that the analytical model of the outage probability fits the result by Monte-Carlo simulation. Moreover, increasing the intensity of active sensors generates an increase of the outage probability. In other words, the system experiences a high outage when the number of the required PRBs by users increases, resulted from the increasing intensity.

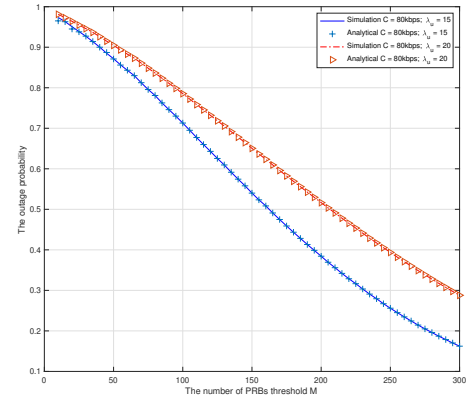


Figure 2. The outage probability in analytical and simulation.

Actually, NB-IoT devices can appear in many different forms: from the indoor devices, such as for gas monitoring and water meters, alerts sensors in building to outdoor devices such as wearable devices for people and animal tracking, health monitoring, the sensors for tracking of attributes of land, pollution, noise, rain in agriculture. Therefore, the signal may be affected by different path-loss coefficients. The indoor sensors always need more PRBs to guarantee a required transmission rate because of high path-loss and poor performance in term of SINR.

To see how eNB's intensity impacts the performance, we plot the outage probability, considering various values of eNB intensities while remaining the active sensor's intensity $\lambda_u = 15$ users/km² in Figure 3. We observe that the outage probability decreases when increasing the eNB's intensity. In other words, if the mean number of sensor users that an eNB has to serve is the same, increasing the number of eNBs leads to a decrease in PRB needed in each eNB.

During resource dimensioning procedure, the eNB starts by defining a target outage probability that can be tolerated for a given service. For a given transmission rate, the number of PRBs is set to ensure that the outage probability never exceeds

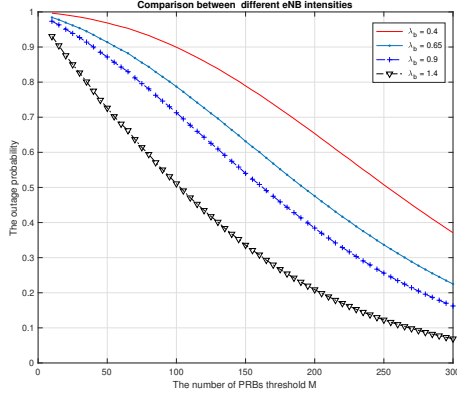


Figure 3. Comparison between different eNB intensities.

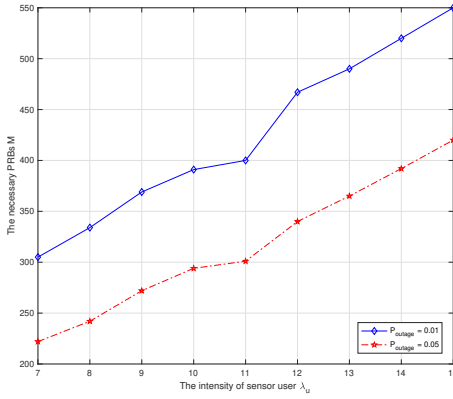


Figure 4. Required PRB M as a function of sensor's intensity.

its target value. Figure 4 presents the number of required PRBs to ensure the target value of the outage probability ($P_{out} = 1\%$ and $P_{out} = 5\%$) with a fixed transmission rate of 80 kbps. We can observe that for the target value of outage probability, the number of resource blocks required in the cell increases when the sensor user intensity increases.

V. CONCLUSION

In this paper, we considered a multi-cell NB-IoT downlink network in which sensor nodes and eNBs are distributed according to a Poisson point process. We have derived an analytical model taking into account the repetition of data transmission to determine the number of required PRBs as a function of sensor user intensity and eNB intensity. The simulation results show that the proposed model is accurate. Moreover, we have established a relationship between the required resources and the sensor users intensity given a target outage probability. In future works, we will investigate the impact of the channel conditions such as shadowing and interference to the network system.

APPENDIX

The moment generating function (i.e., Z-Transform) $f(z)$ of the discrete random variable Λ is calculated as

$$\begin{aligned} f(z) &= \mathbb{E}(z^\Lambda) = \sum_{k=0}^{+\infty} z^k \mathbb{P}(\Lambda = k) \\ &= \prod_{n=1}^N \sum_{k=0}^{+\infty} z^{nk} \mathbb{P}(V_n = k) \end{aligned} \quad (21)$$

Since V_n is a Poisson random variable with parameter w_n , (21) is simplified to

$$f(z) = e^{-\sum_{n=1}^N w_n} e^{\sum_{n=1}^N z^n w_n} \quad (22)$$

By using the definition of x_j in equation (16) and the generating function in equation (13), $f(z)$ in equation (22) becomes

$$\begin{aligned} f(z) &= H e^{\sum_{j=0}^{\infty} z^j \frac{x_j}{j!}} \\ &= H \sum_{p=0}^{+\infty} \frac{z^p}{p!} B_p(x_1, \dots, x_p) \end{aligned} \quad (23)$$

where $H = e^{-\sum_{n=1}^N w_n}$.

Using the definition of Z-Transform and the Taylor expansion of $f(z)$ in 0, we get

$$\mathbb{P}(\Lambda = p) = \frac{H}{p!} B_p(x_1, \dots, x_p) \quad (24)$$

Finally, from the definition of the CCDF, we can obtain

$$\mathbb{P}(\Lambda \geq M) = 1 - \sum_{k=0}^{M-1} \mathbb{P}(\Lambda = k) \quad (25)$$

that leads to the equation (17).

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